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Noncommutative String Theory, the R-Matrix, and Hopf Algebras

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Abstract

Motivated by the form of the noncommutative *-product in a system of open strings and Dp-branes with constant nonzero Neveu-Schwarz 2-form, we define a deformed multiplication operation on a quasitriangular Hopf algebra in terms of its R-matrix, and comment on some of its properties. We show that the noncommutative string theory *-product is a particular example of this multiplication, and comment on other possible Hopf algebraic properties which may underlie the theory.

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1 Introduction

Although the subject of gauge theories in noncommutative geometry is not a new one [1], recently it has enjoyed something of a resurgence. It reappeared in the context of matrix models compactified on tori [2], where it was shown that such models may be reformulated as super Yang-Mills (SYM) theories on noncommutative spacetimes when the Neveu-Schwarz (NS) 2-form B_{ij} is constant and nonzero. This is accomplished by replacing the usual commutative multiplication of functions on the space by a noncommutative one, denoted by *. Subsequent studies dealt with various aspects of noncommutative spaces [3]-[13]. It has also been realised that one gets a noncommutative SYM theory for the case of a system of open strings and Dp-branes in a flat spacetime (with the same condition on B_{ij}), provided that one not only changes all ordinary products to *-products, but also deforms the gauge fields and their transformations [14]. This seems to suggest that noncommutative geometry may be an underlying aspect of a large class of theories.

If this is so, then a mathematically consistent way of dealing with this noncommutativity is needed, and one possibility might be to use the language of Hopf algebras (HAs). A HA extends the notion of an algebra by including information about its representations (through the coproduct), and when it is quasitriangular also gives the algebraic structure of the modules it is represented on through its associated R-matrix, *i.e.* commutation relations between module elements. Perhaps the best-known cases where nontrivial quasitriangular HAs play key roles are quantum groups [15, 16], which may be thought of consisting of matrices whose entries are noncommuting.

It is the intent of this paper to show (in Section 3) that the *-product mentioned above is in fact a particular case of a more general multiplication which may be defined in terms of the R-matrix of a particular quasitriangular HA. This may be a clue that there is indeed a quasitriangular HA structure to these noncommutative theories, and might serve as a tentative first step toward finding that structure. This could in turn lead toward a way of formulating gauge theories on a large class of spaces, not just commutative ones. Some of the implications of such a structure are commented upon in Section 4.

Parts of this work are somewhat pedagogical, but this is because it is intended for an audience for whom the language of HAs may not be too familiar; for those with some knowledge of the subject, Section 2 can be skimmed just to determine the notation we use herein. Others who may be curious about HAs may find the short review useful.

2 Hopf Algebras

This section is meant to be a review of both the formal aspects of what constitutes a Hopf algebra and the explicit example where we consider the algebra of functions over a manifold and the partial derivatives acting on them.

2.1 Formal Definitions

A Hopf algebra $\mathcal{H}$ is an associative algebra with unit 1 over a field k which is also equipped with a counit $\epsilon : \mathcal{H} \to k$, a coproduct (or comultiplication) $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ and an antipode $S : \mathcal{H} \to \mathcal{H}$; the first two of these are defined to be homomorphisms, the third an antihomomorphism, and all three satisfy the relations

$$\begin{aligned} (\Delta \otimes \mathrm{id})\, \Delta(f) &= (\mathrm{id} \otimes \Delta)\, \Delta(f), \\ (\epsilon \otimes \mathrm{id})\, \Delta(f) &= (\mathrm{id} \otimes \epsilon)\, \Delta(f) = f, \\ m\left((S \otimes \mathrm{id})\, \Delta(f)\right) &= m\left((\mathrm{id} \otimes S)\, \Delta(f)\right) = \epsilon(f)1, \end{aligned} \tag{2.1}$$

where $f \in \mathcal{H}$, id is the identity map $f \mapsto f$ and m is the multiplication operation on $\mathcal{H}$ (which will usually be suppressed). For future reference, the first of (2.1) is often called 'coassociativity'.

For clarity, we adopt Sweedler's notation [17], in which we write the coproduct as $\Delta(f) = f_{(1)} \otimes f_{(2)}$, where there is an implied summation. For example, using this convention, the third of (2.1) may be written as $S\left(f_{(1)}\right) f_{(2)} = f_{(1)} S\left(f_{(2)}\right) = \epsilon(f)1$.

We can define the dually paired HA to $\mathcal{H}$, denoted $\mathcal{H}^*$, in the following way: As a vector space over k, $\mathcal{H}^*$ is just the dual space to $\mathcal{H}$, so there is an inner product taking $\mathcal{H}^* \otimes \mathcal{H}$ to k, written as $\langle x, f \rangle$ for $x \in \mathcal{H}^*$ and $f \in \mathcal{H}$. $\mathcal{H}^*$ may be given a Hopf algebra structure by defining the multiplication, unit, coproduct, counit and antipode on $\mathcal{H}^*$ via

$$\begin{aligned} \langle xy, f \rangle &:= \langle x \otimes y, \Delta(f) \rangle, \\ \langle 1, f \rangle &:= \epsilon(f), \\ \langle \Delta(x), f \otimes g \rangle &:= \langle x, fg \rangle, \\ \epsilon(x) &:= \langle x, 1 \rangle, \\ \langle S(x), f \rangle &:= \langle x, S(f) \rangle. \end{aligned} \tag{2.2}$$

It is straightforward to show that these operations satisfy all the HA conditions.

$\mathcal{H}^*$ may be thought of as an algebra of operators on $\mathcal{H}$ when we define the (left) action of $x \in \mathcal{H}^*$ on $f \in \mathcal{H}$, denoted $x \cdot f$, as

$$x \cdot f := f_{(1)} \left\langle x, f_{(2)} \right\rangle. \tag{2.3}$$

Since it is easy to show that $(xy)\cdot f = x\cdot(y\cdot f)$, this is indeed an action of elements of $\mathcal{H}^*$ on those of $\mathcal{H}$, and thus gives a representation of $\mathcal{H}^*$ with $\mathcal{H}$ as the module. Furthermore, there is a sort of Leibniz rule: $x\cdot(fg) = (x_{(1)}\cdot f)(x_{(2)}\cdot g)$. To give $\mathcal{H}^*$ an interpretation, notice that $P_0(x) := x - \epsilon(x)1$ and $P_1(x) := \epsilon(x)1$ are projections on $\mathcal{H}^*$, so $\mathcal{H}^* = \ker\epsilon \oplus k1$. Note that for $x \in \ker\epsilon$, $x\cdot 1 = 0$, so these may be thought of as derivatives on $\mathcal{H}$; elements of $k1$ just multiply elements of $\mathcal{H}$ by elements of k. Finally, if we have a concept of derivatives on $\mathcal{H}$, we can define an integral on $\mathcal{H}$ as a linear map $\int : \mathcal{H} \to k$ such that $\int x\cdot f = 0$ for any $x \in \ker\epsilon$.

A quasitriangular Hopf algebra is an HA $\mathcal{H}$ for which there is a special invertible element $R \in \mathcal{H}\otimes\mathcal{H}$, called the R-matrix, which has the properties

$$\begin{aligned}(\tau\circ\Delta)(f) &= R\,\Delta(f)\,R^{-1},\\ (\Delta\otimes\mathrm{id})(R) &= R_{13}R_{23},\\ (\mathrm{id}\otimes\Delta)(R) &= R_{13}R_{12},\end{aligned} \tag{2.4}$$

where $\tau : f\otimes g \mapsto g\otimes f$ and the subscripts on R in the latter two above tell in which pieces of $\mathcal{H}^{\otimes 3}$ R lives, *i.e.* if $R = r_\alpha \otimes r^\alpha$ (sum implied), then $R_{13} := r_\alpha \otimes 1 \otimes r^\alpha$. One consequence of these properties on the R-matrix is the fact that it must satisfy the Yang-Baxter equation (YBE)

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}. \tag{2.5}$$

(Note that since $R = 1\otimes 1$ satisfies all the above, all HAs are in a sense trivially quasitriangular.)

2.2 Hopf Algebra of Functions and Derivatives

Let $\mathcal{F}$ be the space of functions mapping $\mathbb{R}^{p+1}$ into $\mathbb{C}$. $\mathcal{F}$ is made into a commutative associative algebra in the usual way, *e.g.* $(fg)(x) := f(x)g(x)$ for $f, g \in \mathcal{F}$ and $x \in \mathbb{R}^{p+1}$. Furthermore, $\mathcal{F}$ can be extended to an HA if we define the following on the coordinate maps x^i:

$$\begin{aligned}\Delta\left(x^i\right) &:= x^i\otimes 1 + 1\otimes x^i,\\ \epsilon\left(x^i\right) &:= 0,\\ S\left(x^i\right) &:= -x^i.\end{aligned} \tag{2.6}$$

Once we have the above (as well as the relations $\Delta(1) = 1\otimes 1$, $\epsilon(1) = 1$ and $S(1) = 1$, of course), we can use the fact that Δ and ϵ are homomorphisms and S is an antihomomorphism to extend them to all monomials of the coordinate functions and (ignoring questions of completeness) thus to all of $\mathcal{F}$.

We define $\mathcal{F}^*$, the dually paired HA to $\mathcal{F}$, to be spanned by elements $\{\partial_i | i = 1, \ldots, D\}$, where the inner product between $\partial_i \in \mathcal{F}^*$ and a monomial in $\mathcal{F}$ is

$$\left\langle \partial_i, \prod_{p=1}^{N} \left(x^{j_p}\right)^{n_p} \right\rangle = \sum_{p=1}^{N} \delta_{i j_p} \delta_{n_p 1} \prod_{q \neq p} \delta_{n_q 0}, \tag{2.7}$$

as well as $\langle \partial_i, 1 \rangle = 0$. Once we have this, the HA structure of $\mathcal{F}^*$ is immediate: The form of the coproduct on $\mathcal{F}$ tells us that $\mathcal{F}^*$ is commutative, the commutativity of $\mathcal{F}$ gives $\Delta(\partial_i) = \partial_i \otimes 1 + 1 \otimes \partial_i$, and $\epsilon(\partial_i) = 0$ and $S(\partial_i) = -\partial_i$.

The action of ∂_i on a monomial in $\mathcal{F}$ can be computed as well; it will perhaps be no surprise to the reader that the result is

$$\partial_i \cdot \prod_{p=1}^{N} \left(x^{j_p}\right)^{n_p} = \sum_{p=1}^{N} n_p \left(x^{j_p}\right)^{n_p - 1} \delta_{i j_p} \prod_{q \neq p} \left(x^{j_q}\right)^{n_q}. \tag{2.8}$$

Furthermore, for two arbitrary functions $f(x)$ and $g(x)$,

$$\partial_i \cdot (f(x) g(x)) = (\partial_i \cdot f(x))\, g(x) + f(x)\, (\partial_i \cdot g(x)), \tag{2.9}$$

so $\mathcal{F}^*$ is indeed the space of partial derivatives on functions over $\mathbb{R}^{p+1}$. (Note: For brevity's sake, from now on we will omit the $\cdot$ when speaking of the action of a partial derivative on a function of x.)

3 The *-Product

Motivated by the previously-mentioned literature on noncommuting strings, we now use the formalism just presented to introduce a new *-product on the Hopf algebra $\mathcal{H}$. We examine its properties, and then go on to show that the noncommutative product in string theory is a particular case of this multiplication.

3.1 Formal Definition of the *-Product

Suppose we have two dually paired HAs $\mathcal{H}$ and $\mathcal{H}^*$, and further suppose that there is an R-matrix on $\mathcal{H}^*$. We can thus define a new operation $* : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$ in terms of R and the usual multiplication on $\mathcal{H}$ via

$$f * g := f_{(1)} g_{(1)} \left\langle R_{21}, f_{(2)} \otimes g_{(2)} \right\rangle. \tag{3.1}$$

(If $\mathcal{H}^*$ is trivially quasitriangular, then $f * g = fg$.) This operation is actually associative as a consequence of the YBE. To see this explicitly, we pick $f, g, h \in \mathcal{H}$, and first compute $(f * g) * h$:

$$\begin{aligned}
(f * g) * h &= \left(f_{(1)} g_{(1)} \left\langle R_{21}, f_{(2)} \otimes g_{(2)} \right\rangle\right) * h \\
&= f_{(1)} g_{(1)} h_{(1)} \left\langle R_{21}, f_{(2)} g_{(2)} \otimes h_{(2)} \right\rangle \left\langle R_{21}, f_{(3)} \otimes g_{(3)} \right\rangle \\
&= f_{(1)} g_{(1)} h_{(1)} \left\langle R_{32} R_{31} R_{54}, f_{(2)} \otimes g_{(2)} \otimes h_{(2)} \otimes f_{(3)} \otimes g_{(3)} \right\rangle \\
&= f_{(1)} g_{(1)} h_{(1)} \left\langle R_{32} R_{31} R_{21}, f_{(2)} \otimes g_{(2)} \otimes h_{(2)} \right\rangle.
\end{aligned} \tag{3.2}$$

In going fron the first line to the second, we used the coassociativity of Δ; next, the third of (2.2), the second of (2.4) and a relabelling of the indices so as to stick everything into one inner product; and the last step used the first of (2.2). If we now calculate $f*(g*h)$, an exactly analogous compution gives the result above with the left argument of the inner product replaced by $R_{21}R_{31}R_{32}$, and by using the YBE (with the first and third tensor spaces swapped), we find the two quantities are exactly the same. Hence, $(f*g)*h = f*(g*h)$ and $\widehat{\mathcal{H}}$, the algebra constructed from (the vector space) $\mathcal{H}$ and the *-multiplication is associative. Since it turns out that $1*f = f*1 = f$, $\widehat{\mathcal{H}}$ is unital as well.

This is in general a noncommutative multiplication: Even if $\mathcal{H}$ itself is commutative, $\widehat{\mathcal{H}}$ may not be if $R_{21} \neq R$. To examine this a bit more, it can be shown that the defining relations for R imply that $(\epsilon \otimes \mathrm{id})(R) = (\mathrm{id} \otimes \epsilon)(R) = 1 \otimes 1$, so that we may define a new quantity $\Theta \in \ker\epsilon \otimes \ker\epsilon$ as $\Theta := R - 1\otimes 1$. When written in terms of Θ, the *-product becomes $f*g = fg + f_{(1)}g_{(1)}\left\langle \Theta_{21}, f_{(2)} \otimes g_{(2)}\right\rangle$. Suppose we now define $\Theta \equiv \theta_\alpha \otimes \theta^\alpha$ and compute $\theta^\alpha \cdot (f\,\theta_\alpha \cdot g)$:

$$\begin{aligned}
\theta^\alpha \cdot (f\,\theta_\alpha \cdot g) &= \theta^\alpha \cdot \left(f g_{(1)}\right)\left\langle \theta_\alpha, g_{(2)}\right\rangle \\
&= f_{(1)}g_{(1)}\left\langle \theta^\alpha, f_{(2)}g_{(2)}\right\rangle\left\langle \theta_\alpha, g_{(3)}\right\rangle \\
&= f_{(1)}g_{(1)}\left\langle \Delta\left(\theta^\alpha\right) \otimes \theta_\alpha, f_{(2)} \otimes \Delta\left(g_{(2)}\right)\right\rangle,
\end{aligned} \tag{3.3}$$

where in the last step we have split up $f_{(2)}g_{(2)}$ to get the coproduct of θ^α. The third of (2.4) in terms of Θ is $(\mathrm{id} \otimes \Delta)(\Theta) = \Theta_{13} + \Theta_{12} + \Theta_{13}\Theta_{12}$; if we use this, and then get rid of $\Delta\left(g_{(2)}\right)$ by exchanging it for a multiplication in the $\mathcal{H}^*$ argument of the inner product, we find

$$\theta^\alpha \cdot (f\,\theta_\alpha \cdot g) = f_{(1)}g_{(1)}\left\langle \Theta_{21} + (1 \otimes \kappa)R_{21}, f_{(2)} \otimes g_{(2)}\right\rangle, \tag{3.4}$$

where $\kappa := \theta^\alpha\theta_\alpha$. Let us now suppose that there is a kind of 'tracelessness' condition on Θ, and κ vanishes. We must stress that this is purely an assumption, but if it does in fact hold, then we conclude that

$$f*g = fg + \theta^\alpha \cdot (f\,\theta_\alpha \cdot g), \tag{3.5}$$

in other words, $f*g$ and fg differ by a 'total derivative', since $\theta^\alpha \in \ker\epsilon$. It follows that if there is an integral $\int$ defined over $\mathcal{H}$, $\int f*g = \int fg$.

A comment on the above: The above can be easily extended to the case where we have a $N \times N$ matrix-valued Hopf algebra, *i.e.* $\mathcal{H} \otimes M_N(k)$, in which case the *-product includes matrix multiplication as well: $(f*g)^i{}_j = f^i{}_k * g^k{}_j$. Note that this is *not* the same as a quantum group, since $\mathcal{H}$ is not being thought of as the set of functions over the group manifold. Thus, when we take coproducts *et al.*, the 'matrix part' is unaffected.

Also, the *-product is not unique: If R_{21} is replaced by R^{-1}, the new * is still associative, and the other results also follow with suitable modifications. In the case where the HA is *triangular*, $R_{21} = R^{-1}$ by definition, and the two different *-products coincide.

3.2 Noncommutative String Theory

Recently [14], it has been shown that when one considers open strings and Dp-branes in a space with metric g_{ij} and constant nonvanishing Neveu-Schwarz 2-form B_{ij} $(i,j=0,\ldots,p)$, the theory can be reformulated as a SYM theory on a space where the coordinates no longer commute, but instead satisfy the deformed relation

$$x^i * x^j - x^j * x^i = i\theta^{ij}, \tag{3.6}$$

where

$$\theta^{ij} := -\left(2\pi\alpha'\right)^2 \left(\frac{1}{g+2\pi\alpha' B} B \frac{1}{g-2\pi\alpha' B}\right)^{ij}. \tag{3.7}$$

More generally, it is shown that if $f(x)$ and $g(x)$ are (matrix-valued) functions, the noncommutative product between two functions $f(x)$ and $g(x)$ is

$$f(x) * g(x) := e^{\frac{i}{2}\theta^{ij}\frac{\partial}{\partial\xi^i}\frac{\partial}{\partial\zeta^j}} f(x+\xi)g(x+\zeta)\Big|_{\xi=\zeta=0}. \tag{3.8}$$

The action then can be expressed in SYM form provided that all ordinary multiplications are replaced by this *-product, the 'closed string metric' g_{ij} is replaced by the 'open string metric' $G_{ij} := g_{ij} - \left(2\pi\alpha'\right)^2 \left(Bg^{-1}B\right)_{ij}$ and the gauge field A_i is replaced by $\widehat{A}_i$, which depends on both A_i (and its derivatives) and θ^{ij}.

If we compare (3.8) to (3.1), it suggests that a candidate for the R-matrix is

$$R = e^{-\frac{i}{2}\theta^{ij}\partial_i\otimes\partial_j}, \tag{3.9}$$

but we must make sure it satisfies all the appropriate relations. Since $\mathcal{F}^*$ is commutative, and $\Delta\left(\partial_i\right)$ is symmetric, it is evident that the first of (2.4) is satisfied. To check the second of (2.4), note that

$$\Delta\left(\prod_{\ell=1}^{k}\partial_{i_\ell}\right) = \sum_\sigma\sum_{\ell\le k}\frac{1}{\ell!(k-\ell)!}\partial_{i_{\sigma(1)}}\ldots\partial_{i_{\sigma(\ell)}}\otimes\partial_{i_{\sigma(\ell+1)}}\ldots\partial_{i_{\sigma(k)}}, \tag{3.10}$$

where σ is a permutation of $(1,\ldots,k)$. Therefore,

$$\begin{aligned}\left(\Delta\otimes\mathrm{id}\right)(R) &= \sum_{k=0}^{\infty}\frac{(-i)^k}{2^k k!}\left(\prod_{p\le k}\theta^{i_p j_p}\right)\Delta\left(\prod_{\ell\le k}\partial_{i_\ell}\right)\otimes\left(\prod_{q\le k}\partial_{j_q}\right)\\ &= \sum_{k=0}^{\infty}\sum_{\ell\le k}\frac{(-i)^k}{2^k k!\ell!(k-\ell)!}\left(\prod_{p\le k}\theta^{i_p j_p}\right)\sum_\sigma\left(\prod_{m=1}^{\ell}\partial_{i_{\sigma(m)}}\right.\\ &\quad\left.\otimes\prod_{n=\ell+1}^{k}\partial_{i_{\sigma(n)}}\right)\otimes\left(\prod_{q\le k}\partial_{j_q}\right)\end{aligned}$$

$$= \sum_{\ell=0}^{\infty} \frac{(-i)^{\ell}}{2^{\ell}\ell!} \left(\prod_{p\leq \ell} \theta^{i_p j_p} \partial_{i_p} \right) \otimes \left[\sum_{k\geq \ell} \frac{(-i)^{k-\ell}}{2^{k-\ell}(k-\ell)!} \right.$$

$$\left. \left(\prod_{n=\ell+1}^{k} \theta^{i_n j_n} \partial_{i_n} \right) \otimes \left(\prod_{q\leq k} \partial_{j_q} \right) \right], \tag{3.11}$$

where in the last step we have switched the sums over k and ℓ, and also used the commutativity of the partial derivatives in the third tensor space to get rid of σ (picking up a $k!$ in the process). If we now split the product in the third space into two, one from 1 to ℓ and the other from $\ell+1$ to k, and let $k-\ell = r$, then we get

$$\begin{aligned}
(\Delta \otimes \mathrm{id})(R) &= \sum_{\ell,r=0}^{\infty} \frac{(-i)^{\ell+r}}{2^{\ell+r}\ell! r!} \left(\prod_{p\leq \ell} \theta^{i_p j_p} \partial_{i_p} \right) \otimes \left(\prod_{n\leq r} \theta^{i'_n j'_n} \partial_{i'_n} \right) \\
&\quad \otimes \left(\prod_{q\leq \ell} \partial_{j_q} \prod_{q'\leq r} \partial_{j'_{q'}} \right) \\
&= \left[\sum_{\ell=0}^{\infty} \frac{(-i)^{\ell}}{2^{\ell}\ell!} \left(\theta^{ij} \partial_i \otimes 1 \otimes \partial_j \right)^{\ell} \right] \left[\sum_{r=0}^{\infty} \frac{(-i)^{r}}{2^{r} r!} \left(1 \otimes \theta^{i'j'} \partial_{i'} \otimes \partial_{j'} \right)^{r} \right] \\
&= R_{13} R_{23}.
\end{aligned} \tag{3.12}$$

A very similar calculation confirms that the last of (2.4) holds as well, so (3.9) is in fact an R-matrix, and $\mathcal{F}$ a quasitriangular HA (in fact, since $R_{21} = R^{-1}$, it is triangular). The YBE is therefore satisfied by this R, and thus * is associative, as proven above.

To check that this R-matrix gives us the correct commutation relation (3.6), we simply compute $x^i * x^j$:

$$\begin{aligned}
x^i * x^j &= \left(x^i\right)_{(1)} \left(x^j\right)_{(1)} \left\langle R_{21}, \left(x^i\right)_{(2)} \otimes \left(x^j\right)_{(2)} \right\rangle \\
&= x^i x^j \left\langle R_{21}, 1 \otimes 1 \right\rangle + x^i \left\langle R_{21}, 1 \otimes x^j \right\rangle + x^j \left\langle R_{21}, x^i \otimes 1 \right\rangle \\
&\quad + \left\langle R_{21}, x^i \otimes x^j \right\rangle \\
&= x^i x^j + \frac{i}{2} \theta^{ij}.
\end{aligned} \tag{3.13}$$

Therefore, by switching i and j and subtracting, we recover (3.6). (3.8) holds almost by definition, since, from (3.1), $f * g$ is simply the product between the action of the first tensor space of R_{21} on f and the action of the second on g.

This R also satisfies the 'tracelessness' condition: If we subtract $1 \otimes 1$ from R_{21} and multiply the two tensor product spaces together, we immediately get κ, which is evidently zero due to the asymmetry (and constancy) of θ^{ij} and commutativity of the partial derivatives, so $f * g$ and fg differ by a total derivative. And since the original multiplication was commutative to begin with, $\int \mathrm{tr}\,(f * g) = \int \mathrm{tr}\,(g * f)$.

4 Noncommutative Gauge Theories

We now make some comments on noncommutative gauge theories and how they may or may not relate to HAs.

4.1 Algebraic Structure

In Section 2.2, we showed that for the commutative case, there is a HA structure to both functions and derivatives on $\mathbb{R}^{p+1}$. However, even though the noncommutative $\widehat{\mathcal{F}}$ is an associative unital algebra, it is *not* a HA, as can be seen from the following: As we proved, the unit in $\mathcal{F}$ is the unit in $\widehat{\mathcal{F}}$, so that if $\widehat{\mathcal{F}}$ is a HA and has a counit $\widehat{\epsilon}$, then the fact that this counit is a homomorphism from $\widehat{\mathcal{F}}$ to $\mathbb{C}$ means $\widehat{\epsilon}(1) = 1$. Now, consider $x^i * x^j - x^j * x^i$; the counit will map this to zero. But this commutator is $i\theta^{ij}1$, so we have a contradiction. Therefore, $\widehat{\mathcal{F}}$ cannot be a HA.

This is no real surprise. In the first place, the requirements necessary for a space to be a HA are very restrictive, and in general there is no reason to expect an arbitrary algebra to also be a HA. Furthermore, although $\widehat{\mathcal{F}}^*$ has a coalgebra structure (a counit and a coproduct) due to the fact that $\widehat{\mathcal{F}}$ is unital and associative, there is no 'deformed derivative'. We can certainly define an inner product between the two spaces, but due to the lack of a coproduct on $\widehat{\mathcal{F}}$, there is no action of $\widehat{\mathcal{F}}^*$ on it, and thus no concept of derivative. This is borne out by the fact that we must use the ordinary partial derivative to define the noncommutative SYM field strength, via $\widehat{F}_{ij} = \partial_{[i}\widehat{A}_{j]} - i\widehat{A}_{[i} * \widehat{A}_{j]}$ [14]; if a deformed derivative were available, it would be the more natural choice, but this is not the case.

However, although there is no interpretation of elements of $\widehat{\mathcal{F}}^*$ as objects with a *local* action on $\widehat{\mathcal{F}}$, it might still be possible to interpret them as *nonlocal* operators. As an illustration of this, consider the case of the 2-dimensional quantum hyperplane: The coordinates x, y generate an algebra (the functions on the plane) modulo the commutation relation $xy = qyx$, and the 'derivatives' ∂_x, ∂_y act on a function $f(x,y)$ (ordered so that all xs appear to the left of all ys) as

$$\begin{aligned} \partial_x f(x,y) &= \frac{f(q^{-2}x, y) - f(x,y)}{(q^{-2}-1)\,x}, \\ \partial_y f(x,y) &= \frac{f(q^{-2}x, q^{-2}y) - f(q^{-2}x, y)}{(q^{-2}-1)\,y}, \end{aligned} \tag{4.1}$$

where $q \in \mathbb{R}$ [18]. Note that as $q \to 1$, these become ordinary derivatives, but otherwise they are nonlocal difference operators. The string case could be similar, with θ^{ij} playing the role of $\ln q$. This granularisation of the spacetime might explain the absence of small instanton singularities [19, 14] in the noncommuting theory, by smearing out such objects over more than one point.

4.2 Gauge Fields and Hopf Algebras

The fact that the *-product can be defined in terms of an abstract HA and includes the noncommutative string theory case hints at the possibility of describing the entire theory using a quasitriangular HA, where the R-matrix depends on θ^{ij}. However, this is certainly not sufficient, since we have not considered the gauge field $\widehat{A}_i$. We have also not addressed the matter of the map $A_i \mapsto \widehat{A}_i$ which allow us to cast the action in SYM form. We now address both of these issues.

θ^{ij} is inherently a HA parameter; it appears in the R-matrix and therefore describes the HA structure of $\mathcal{F}$ and $\mathcal{F}^*$. This can be seen either explicitly, as in the relation $(\tau \circ \Delta)(x) = R\Delta(x)R^{-1}$ on any $\mathcal{H}^*$, or via the commutation relations of elements of $\mathcal{H}$, which may be expressed as

$$gf = \left\langle R, f_{(1)} \otimes g_{(1)} \right\rangle \, f_{(2)} g_{(2)} \, \left\langle R^{-1}, f_{(3)} \otimes g_{(3)} \right\rangle . \tag{4.2}$$

So, motivated by these facts, it therefore seems reasonable to conjecture that *all* θ-dependence in the theory is in the HA structure of $\mathcal{F}$ and $\mathcal{F}^*$.

If this is true, then the θ-dependence in the noncommuting theory must arise from the underlying HA describing the commutative theory. This gives us a bit of information about the gauge fields: We know that the change of variables between the gauge fields of the two theories involves θ [14], which means some sort of HA-derived operation is involved in going from one to the other. Since, as we proved in the previous section, the noncommutative algebra $\widehat{\mathcal{F}}$ is not a HA, there must be some element $W \in \mathcal{F}$ with given HA properties (coproduct, *etc.*) which is related to both A_i and $\widehat{A}_i$. The assumption that all θ-dependence is in the HA structure and not in $\mathcal{F}$ (as a vector space) itself leads to the conclusion that W is independent of θ, and thus must be related to A_i, since this is the gauge field for the $\theta^{ij} = 0$ case. If we also assume that the only dependence on the open string metric G_{ij} is from constructing Lorentz-invariant quantities in the integral, *e.g.* $\int \mathrm{d}^{p+1}x \sqrt{G} G^{ij} \alpha_i \beta_j$, then W should also be G-independent.

At this writing, we do not know precisely what this element might be, but a natural candidate would be the Wilson line (which explains why we call it W): Recall that the Wilson line $W_{C(x_0,x)}$ is given by

$$W_{C(x_0,x)} := \mathrm{P} e^{i \int_{C(x_0,x)} A}, \tag{4.3}$$

where $C\,(x_0, x)$ is a path going from x_0 to x. It depends on the gauge field of the commutative theory, and is independent of θ and G, so it fits the criteria we just outlined. We might also be able to obtain $\widehat{A}_i$ in the following way: Our proposed W must relate both gauge fields in a HA-dependent way, but be independent of θ. The relation of W_C to A_i is through the multiplication on $\mathcal{F}$, via the path-ordered exponent. We could therefore *define* $\widehat{A}_i$ to be that function for which the same W_C can be written as a path-ordered exponential as well, but this time using the *-product instead (denoted $\widehat{e}$). In other words,

$$\begin{aligned} W_{C(x_0,x)} &= \mathrm{P}\widehat{e}^{\,i\int_{C(x_0,x)}\widehat{A}} \\ &= 1 + i\int_{C(x_0,x)}\widehat{A} - \int_{C(x_0,x)}\left(\widehat{A} * \int_{C(x_0,x_1)}\widehat{A}\right) + \ldots \end{aligned} \tag{4.4}$$

Then the condition

$$\frac{\partial}{\partial\theta^{ij}}\mathrm{P}\widehat{e}^{\,i\int_{C(x_0,x)}\widehat{A}} = 0 \tag{4.5}$$

would give a differential equation involving $\widehat{A}_i$, θ^{ij} and W_C, which we could presumably solve to find $\widehat{A}_i$ as a function of θ^{ij} and (derivatives of) A_i. Our first calculations show enough similarities to the first of Equation (3.5) of [14] to be encouraging.

As for the gauge transformation, the same idea applies: A finite transformation on W_C given by a unitary matrix $U(x) = e^{i\lambda(x)}$ gives $U(x)W_{C(x_0,x)}U^{-1}(x_0)$. This should be the same as if we started with W_C in terms of * and $\widehat{A}_i$, and then transformed by U expressed in terms of * and a new matrix $\widehat{\lambda}$ via $U = \widehat{e}^{i\widehat{\lambda}}$. Then by solving

$$\frac{\partial}{\partial\theta^{ij}}\left[\widehat{e}^{\,i\widehat{\lambda}(x)} * \widehat{e}^{\,i\int_{C(x_0,x)}\widehat{A}} * \widehat{e}^{\,-i\widehat{\lambda}(x_0)}\right] = 0, \tag{4.6}$$

we would obtain the second of (3.5) in [14].

So the sought-after $W \in \mathcal{F}$ could be related to the Wilson loop W_C, even though we do not claim to have offered anything more than a vague justification for this feeling. We have not said anything about the HA properties of W (although the HA enters in going from e to $\widehat{e}$, via *), nor have we said how the curve $C(x_0, x)$ would be chosen, since we have taken it to be completely arbitrary. And perhaps most importantly, we have not considered what the action might be as a function of W (and R) such that we ultimately end up with a SYM form when we go to the noncommuting spacetime. Regardless, there is enough evidence to consider our Wilson line guess as reasonable.

Everything we have done above has been for the specific case of a noncommutative string, where we started with a commutative space ($\mathbb{R}^{p+1}$); formulating it in the HA language as we propose would also be a way of coming up with gauge theories where the original space might be noncommutative. Steps in this direction have been made when the gauge group (and possibly also the spacetime) is deformed [20], and this may mean there is some hope of success for the present problem.

5 Conclusions

We have shown that the product appearing in noncommutative string theory is simply a specific case of one which may be defined in terms of a quasitriangular HA. This fact has lead us to speculate that it may be possible to relate an arbitrary noncommutative gauge theory to a quasitriangular HA in this way, and we have commented on some ways in which this may be done. We hope to address this possibility in future work.

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